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Testing uniformity on the circle using spacings when data are rounded

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ABSTRACT

Testing for uniformity for any given data set on the circle is an important first step before any further inference. One important class of tests are those based on spacings, which assume that the data are measured on a continuous scale. In practice however, the observed data may come grouped, or the recorded observations may be rounded values. Ignoring this fact can result in incorrect Type I error probabilities and inference, especially if the degree of rounding is severe or if the sample size is large. In this article, we propose a simple modification to such rounded data, which then allows us to continue to use the Rao's spacing test and its exact critical values, without affecting the probability of Type I error. We provide theoretical justification for the suggested modification, as well as simulation studies that demonstrate its strong and robust performance.

1. Introduction

Circular data arise when measurements are directions on the 2-dimensional plane. Such directional measurements can be represented as points on the circumference of a unit circle, or equivalently, as angles on the $[0°, 360°)$ scale with a conveniently chosen zero direction and a sense of rotation (clockwise or anti-clockwise). Examples abound from diverse fields such as biology (vanishing directions of homing pigeons), meteorology (wind directions), geology (directions of sediment deposit), materials science (orientations of lattice in crystals), etc. A circular framework can also be used to represent observations on a cyclical phenomenon with a known period. Here, a point on the circle represents the position of the observation relative to the entire cycle. Some examples are in medicine (the time of day when cortisol level reaches its peak), sociology (time of death in the year relative to one's birthday), resource planning (arrival time at an emergency room), etc.

Since a point on the circle can have different representations depending on the choice of zero direction and/or the sense of rotation, circular observations need inferential techniques that are *invariant* to these choices. In fact, the spacings i.e. the arc-lengths between successive observations, form a "maximal invariant" (with respect to the rotation group of transformations) in this context and play an important role. Several monographs provide detailed treatment of circular data and their analysis — see for example Batschelet (1981), Fisher (1995), Jammalamadaka and SenGupta (2001) and Mardia and Jupp (2009). There is also a growing body of literature on the topic of circular statistics, to demonstrate which we just cite two articles: Ghosh et al. (1999) for detecting change-points and Basu and Jammalamadaka (2002) for testing unimodality.

Often, the first step in dealing with such circular observations is to test whether they are uniformly distributed around the circle, in which case there is no well-defined mean-direction. For example, when a batch of homing pigeons is released, do they have a

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preferred direction of flight, or are they randomly flying in all directions when they vanish over the horizon? Or, in the case of emergency room arrivals, are the arrivals equally likely to occur during any time of the day, or are certain times more likely? Due to the reasons mentioned earlier, classical tests of uniformity do not fulfill this invariance property and several modified procedures that are invariant, are available.

One such test for circular uniformity is the so-called Rayleigh's test (Rayleigh, 1880), which is based on the length of the resultant vector, when each observation is treated as a unit vector. Rayleigh's test however suffers from a serious limitation — it is only useful for testing uniformity against unimodal alternatives.

An alternative choice is to use tests based on spacings. Such tests measure the discrepancy between the observed spacings and their corresponding expected values, which turn out to be $360/n$ under the null-hypothesis of uniformity, when there are n observations and the directions are measured in degrees. Unlike the Rayleigh's test, spacings-based tests of uniformity are valid against any alternative distribution and are thus universally applicable.

Formally, suppose $\theta_1, \ldots, \theta_n$ are n circular observations measured in degrees on the $[0, 360)$ scale. Let $\theta_{(1)} < \theta_{(2)} < \cdots < \theta_{(n)}$ be the corresponding "ordered" values, going either clockwise or anti-clockwise. We define the circular spacings as the gaps between successive observations, namely

$$W_i = \begin{cases} \theta_{(i)} - \theta_{(i-1)}, & i = 2, \ldots, n \\ 360 - \theta_{(n)} + \theta_{(1)}, & i = 1. \end{cases}$$

Depending on the measure of discrepancy used, we can get different members of the family of spacings-based tests. The most popular choice in this family is the Rao's spacing test (Rao, 1969, 1976), defined as

$$R = \frac{1}{2} \sum_{i=1}^{n} \left| W_i - \frac{360}{n} \right|. \tag{1}$$

It calculates the absolute value of the distance between the observed spacings and their corresponding expected values under the assumption of uniformity. The popularity stems partly from the fact that its exact distribution (and corresponding large-sample approximation) under the null-hypothesis of uniformity are well-known. Availability of tables of its critical values (see Russell and Levitin, 1995) and an R routine `rao.spacing.test` in `library(circular)` implementing this test make it easy to use by practitioners. Another notable spacings-based test for uniformity is the Greenwood's test (Greenwood, 1946), defined as

$$G = \sum_{i=1}^{n} \left(W_i - \frac{360}{n} \right)^2. \tag{2}$$

However, a closed-form exact distribution for small samples is not available and there is only limited availability of tables of critical values (see Burrows, 1979; Currie, 1981; Stephens, 1981). Additionally, the lack of software routines makes it less popular than the Rao's spacing test.

In practice, the directional measurements are often reported after some rounding or truncation. In rounding, the data are grouped into class-intervals. A value falling in an interval is represented by the lower or upper endpoint of the interval, depending on its proximity to the respective endpoint. The amount of rounding can vary depending on various factors, such as the problem at hand, the precision of the recording instrument, and the convention in the field of application. For example, angular measurements in the $[0°, 360°)$ continuous scale may be rounded to the nearest multiple of $10°$ for convenience. Here the data are grouped into 36 class-intervals of width $10°$ each, with values falling in the $[5°, 15°)$ range rounded to $10°$, values in $[15°, 25°)$ rounded to $20°$, and finally, proceeding similarly, values in the interval $[355°, 359°) \cup [0°, 5°)$ rounded to $0°$. The reported measurements after rounding are in the set $\{0°, 10°, \ldots, 350°\}$.

Alternatively, the measurements can also be truncated (i.e. rounded down). In this case, the values in an interval are represented by the lower end of that interval. For example, values in $[0°, 10°)$ are represented by $0°$, those in $[10°, 20°)$ represented by $10°$, and so on. Here again, the reported measurements are in the set $\{0°, 10°, \ldots, 350°\}$. One can similarly have upward rounding of data.

Since rounding (and similarly truncation) results in discretization of the data, it leads to spacings with a positive mass at zero. Any spacings-based test statistic, such as the Rao's spacing statistic calculated from rounded data would have a different distribution than the original distribution of the statistic (which is calculated from continuous unrounded data) under the null hypothesis of uniformity. Practitioners often disregard this aspect when they use Rao's spacing test to test for uniformity based on rounded data. Consequently, inference from such rounded data may be incorrect, especially if the degree of rounding is quite high, i.e., with fewer class-intervals used to group the data.

One way to address this issue would be to calculate the distribution of the test statistic when it is based on rounded or truncated data. However spacings tests may not be useful in these contexts because rounding and/or truncating leads to a large number of zero-valued spacings, depending on the degree of rounding. Tests that are not based on spacings (such as a test for discrete uniformity) are not appropriate either, since spacings are maximal invariant in the circular context.

In this article, we propose an alternative method that allows us to continue using the original set of tabulated and readily available critical values. The proposed method assumes that the mechanism of rounding or truncation is known and one can use this knowledge to modify the rounded data. Under uniformity, the test statistic based on this modified rounded data is shown to have the same distribution as the test statistic based on the original unrounded data. This allows the practitioner to continue using the currently available tables of Rao's spacing statistic.

The rest of this article is as follows. In Section 2, we present the details and theoretical justification of the proposed method. In Section 3, we present results of some simulation studies to support our theoretical findings, and in Section 4, we provide a real data example. We end the article with some concluding remarks in Section 5.

2. Proposed method

Suppose θ denotes a continuous circular measurement on the $[0, 360)$ scale. Instead of recording the value of θ, we record a value θ^*, which is a rounded version of θ.

We assume that our rounding scheme uses k possible consecutive values $\alpha_{(1)} < \alpha_{(2)} < \cdots < \alpha_{(k)}$ which are spaced equally apart on the unit circle, where $\alpha_{(1)} \geq 0$ and $\alpha_{(k)} < 360$. Hence, $\alpha_{(2)} - \alpha_{(1)} = \alpha_{(3)} - \alpha_{(2)} = \cdots = \alpha_{(k)} - \alpha_{(k-1)} = 360 - \alpha_{(k)} + \alpha_{(1)} = \frac{360}{k}$. We assume that for any value θ, the corresponding rounded value θ^* will be given by

$$\theta^* = \alpha_{(i)} \text{ if } \theta \in \left[\alpha_{(i)} - \frac{360}{2k},\ \alpha_{(i)} + \frac{360}{2k}\right).$$

Clearly, the choice of k decides the level of rounding. A smaller value of k represents higher degree of rounding with more pronounced rounding effect, and a worsening performance of spacings tests based on such rounded data.

In what follows, we will use the notation "CU" to denote the Circular Uniform distribution.

Theorem 1. *If $\theta \sim$ CU, then $\theta^* \sim$ Discrete Unif$\{\alpha_{(1)}, \ldots, \alpha_{(k)}\}$.*

Proof. Suppose $\theta \sim$ CU. Then for $i = 1, \ldots, k$,

$$\begin{aligned} P(\theta^* = \alpha_{(i)}) &= P\left(\theta \in \left[\alpha_{(i)} - \frac{360}{2k},\ \alpha_{(i)} + \frac{360}{2k}\right)\right) \\ &= \frac{2 \times \frac{360}{2k}}{360} \\ &= \frac{1}{k}. \end{aligned}$$

Hence, $\theta^* \sim$ Discrete Unif$\{\alpha_{(1)}, \ldots, \alpha_{(k)}\}$. □

Given a rounded value θ^*, we propose to obtain a pseudo-value $\tilde{\theta}$ (which we call the modified version of θ^*) as

$$\tilde{\theta} \sim \text{Unif}\left(\theta^* - \frac{360}{2k},\ \theta^* + \frac{360}{2k}\right). \tag{3}$$

Theorem 2. *If $\theta \sim$ CU, then $\tilde{\theta} \sim$ CU.*

Proof. We know

$$\tilde{\theta}|\theta^* \sim \text{Unif}\left(\theta^* - \frac{360}{2k},\ \theta^* + \frac{360}{2k}\right).$$

Also, since $\theta \sim$ CU, we have

$$\theta^* \sim \text{Discrete Unif}\{\alpha_{(1)}, \ldots, \alpha_{(k)}\}.$$

Combining, we get

$$\tilde{\theta} \sim \sum_{i=1}^{k} \frac{1}{k} \text{Unif}\left(\alpha_{(i)} - \frac{360}{2k},\ \alpha_{(i)} - \frac{360}{2k}\right) = \text{CU}. \quad \square$$

Hence, given a set of observations $\theta_1^*, \theta_2^*, \ldots, \theta_n^*$ (which are rounded versions of unobserved values $\theta_1, \ldots, \theta_n$ — a random sample of size n from CU), we can get a random sample of pseudo-values $\tilde{\theta}_1, \tilde{\theta}_2, \ldots, \tilde{\theta}_n$ from CU according to (3). Testing for circular uniformity of the distribution of the unobserved θ-values is equivalent to testing for circular uniformity of the distribution of $\tilde{\theta}$-values. Rao's spacing test based on these unrounded $\tilde{\theta}$-values can now be computed as before.

Let $\tilde{\theta}_{(1)} < \tilde{\theta}_{(2)} < \tilde{\theta}_{(n)}$ denote the ordered pseudo-values. Define the resulting spacings by

$$\tilde{W}_i = \begin{cases} \tilde{\theta}_{(i)} - \tilde{\theta}_{(i-1)}, & i = 2, \ldots, n \\ 360 - \tilde{\theta}_{(n)} + \tilde{\theta}_{(1)}, & i = 1. \end{cases}$$

Then the statistic

$$\tilde{R} = \frac{1}{2} \sum_{i=1}^{n} \left| \tilde{W}_i - \frac{360}{n} \right| \tag{4}$$

has the same distribution as the Rao's spacing statistic based on a random sample of size n from CU. "Large" values of $\tilde{R}$ indicate departure of the $\tilde{\theta}$ values from circular uniformity, which in turn indicates departure from circular uniformity of the underlying distribution of the unobserved θs. This can be done using tabulated critical values of the Rao's spacings test. For future reference, we will denote the Rao's spacing test computed from (unmodified) rounded data by

$$R^* = \frac{1}{2} \sum_{i=1}^{n} \left| W_i^* - \frac{360}{n} \right|. \tag{5}$$

Table 1
Simulated Type I error probabilities of Rao's spacing test for uniformity based on (i) unrounded data: R, (ii) rounded data without adjustment: R^*, and (iii) rounded data with adjustment: $\tilde{R}$. Here n = sample size and k = number of groups when rounding. Results based on 10 000 simulations for each combination of n and k. The nominal Type I error level is $\alpha = 0.05$.

n	k	R	R^*	$\tilde{R}$	n	k	R	R^*	$\tilde{R}$
	5	0.05	0.09	0.05		5	0.05	1.00	0.05
	10	0.05	0.03	0.05		10	0.05	0.15	0.05
5	20	0.05	0.04	0.05	10	20	0.05	0.07	0.05
	36	0.05	0.05	0.05		36	0.05	0.06	0.05
	180	0.05	0.05	0.05		180	0.05	0.05	0.05
	360	0.05	0.05	0.05		360	0.05	0.05	0.05
	5	0.05	1.00	0.05		5	0.05	1.00	0.05
	10	0.05	1.00	0.05		10	0.05	1.00	0.05
20	20	0.05	0.17	0.05	50	20	0.05	1.00	0.05
	36	0.05	0.09	0.05		36	0.05	0.75	0.05
	180	0.05	0.06	0.05		180	0.05	0.06	0.05
	360	0.05	0.05	0.05		360	0.05	0.05	0.05
	5	0.05	1.00	0.05		5	0.05	1.00	0.05
	10	0.05	1.00	0.05		10	0.05	1.00	0.05
100	20	0.05	1.00	0.05	200	20	0.05	1.00	0.05
	36	0.05	1.00	0.05		36	0.05	1.00	0.05
	180	0.05	0.11	0.05		180	0.05	0.46	0.05
	360	0.05	0.07	0.05		360	0.05	0.14	0.05
	5	0.06	1.00	0.05		5	0.05	1.00	0.05
	10	0.05	1.00	0.05		10	0.05	1.00	0.05
500	20	0.05	1.00	0.05	1000	20	0.05	1.00	0.05
	36	0.05	1.00	0.04		36	0.05	1.00	0.05
	180	0.04	1.00	0.05		180	0.05	1.00	0.05
	360	0.05	1.00	0.05		360	0.05	1.00	0.04

Remark 1. The results presented in this section for "rounded" data carry over equally well to the case of "truncated" data, with corresponding modifications. In that case, the truncated value corresponding to θ would be given by

$$\theta^* = \alpha_{(i)} \text{ if } \theta \in \left[\alpha_{(i)},\ \alpha_{(i+1)}\right)$$

and the pseudo-value (the adjusted value after correcting for the effect of truncation) $\tilde{\theta}$ would be given by

$$\tilde{\theta}|\theta^* \sim \text{ Unif}\left(\theta^*,\ \theta^* + \frac{360}{k}\right).$$

3. Simulation studies

An extensive simulation study was conducted to compare the performances of following three versions of Rao's spacing test under the null hypothesis of uniformity:

(i) the test R based on the original data before rounding, as defined in (1),
(ii) the test R^* based on rounded data, as defined in (5), and
(iii) the test $\tilde{R}$ based on the data obtained after applying the proposed modification to remove the rounding effect, as defined in (4).

3.1. Comparison of Type I errors

We considered various combinations of sample size (n) and number of classes (k) for rounding. For each case, the nominal Type I error level was chosen to be $\alpha = 0.05$ and number of simulations was set at 10,000. Table 1 gives the simulated Type I error probabilities for the three statistics for selected combinations of n and k. The 5% critical values were obtained using the exact distribution of the statistic R under the null hypothesis of uniformity.

From Table 1, we see that the statistic R^*, which is based on the rounded data, fails to maintain the nominal Type I error level. As expected, for a fixed sample size, the Type I error of R^* inflates as k decreases (i.e. as the degree of rounding gets more severe). On the other hand, for a fixed degree of rounding, the Type I error of R^* gets larger as the sample size increases. Thus, even a minor amount of rounding will have a severe impact on Type I error if the sample size is large enough. For example, when the measurements are rounded to the nearest degree (i.e., $k = 360$), there is a 14% chance of incorrectly rejecting the null hypothesis of uniformity based on a sample of size 200. However, the statistic $\tilde{R}$ based on modified rounded data continues to maintain the Type I error probability, irrespective of the sample size and the degree of rounding.

Fig. 1 plots the probability of Type I error for Rao's test against the sample size when observations are rounded to the nearest multiple of 10°, resulting in 36 possible values after rounding.

Simulation results in case of the Greenwood's statistic were similar to those for the Rao's spacing test. Because of space considerations, they are not being presented here.

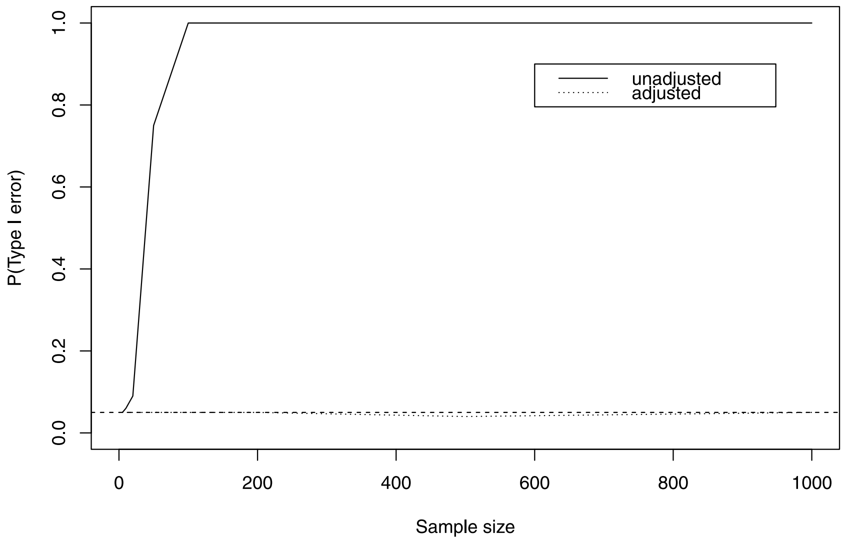

Fig. 1. Plot of Type I error probability against sample size for Rao's spacing test for uniformity. Solid line: based on rounded data without adjustment (R^*), dotted line: based on rounded data with adjustment ($\tilde{R}$). Here, observations are rounded to the nearest multiple of 10°, resulting in $k = 36$ groups. The nominal Type I error level is $\alpha = 0.05$, shown using the dashed line. Results based on 10,000 simulated values.

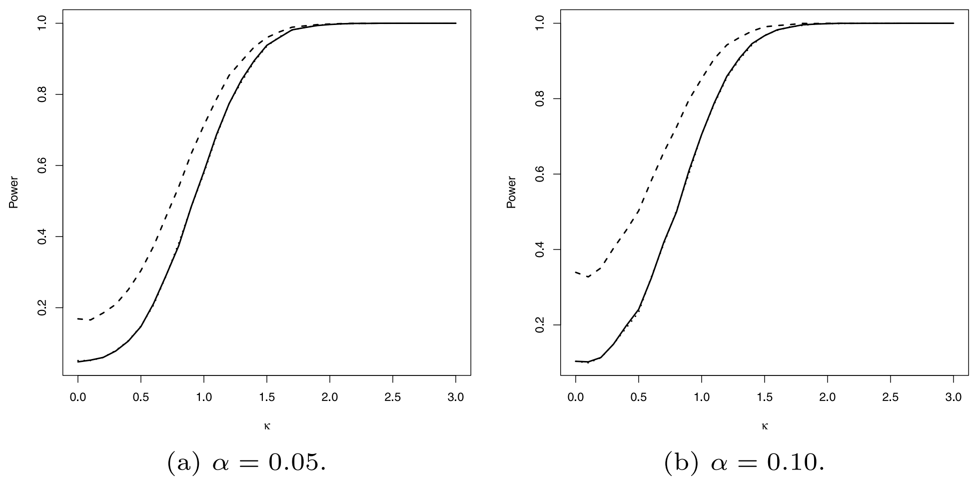

Fig. 2. Power comparison of the 3 versions of the Rao's spacings test based on samples from the von Mises $(0, \kappa)$ distribution. Sample size $n = 30$, number of groups $k = 36$, P(Type I error) $= \alpha$. Fig. 2(a): $\alpha = 0.05$, Fig. 2(b): $\alpha = 0.10$. Solid line: R, dashed line: R^*, dotted line: $\tilde{R}$. Results based on 10,000 simulated values.

3.2. Power comparison

Next, we conducted simulation studies to compare the power curves of the R, R^* and $\tilde{R}$ tests. Random samples of size $n = 40$ were repeatedly selected from the von Mises$(0, \kappa)$ distribution. For each sample, the values of R, R^*, and $\tilde{R}$ were then calculated. Rounded data were obtained by rounding each observation to the nearest multiple of 10° (i.e., $k = 36$). This was repeated 10,000 times and the proportion of times each statistic exceeded the upper α quantile of the corresponding Rao's spacing statistic was noted.

Note that, for the von Mises distribution, the parameter κ, known as the concentration parameter, controls its degree of non-uniformity. The uniform distribution is obtained when $\kappa = 0$ and values of κ further away from zero lead to a more non-uniform distribution. The power curves were obtained by varying $\kappa \in 0(0.1)3$.

The resulting power curves are presented in Fig. 2. Although the test based on unmodified rounded data (R^*) has higher power, it fails to maintain the nominal Type I error level. However, the test based on rounded data with the proposed modification ($\tilde{R}$) performs identical to that based on unrounded data and is thus preferable.

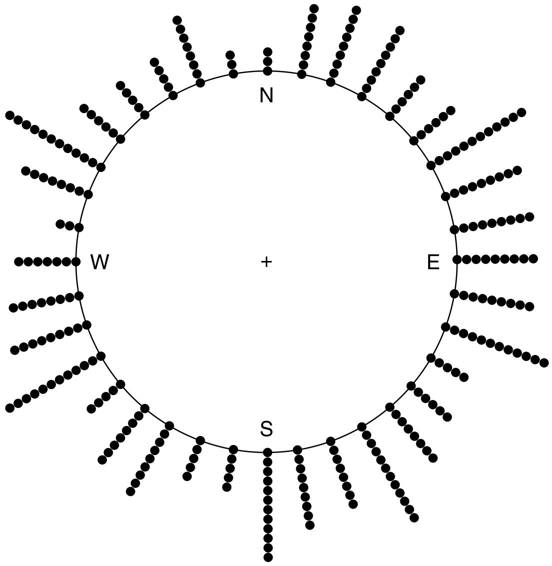

Fig. 3. Dance directions of 279 honeybees viewing a zenith patch of artificially polarized light. Data from Appendix B.9 on page 244 of Fisher (1995).

4. Data example

Wehner and Strasser (1985) describes an experiment conducted to demonstrate that specialized photoreceptors at the dorsal margin of the eyes of honeybees are necessary for detecting polarized skylight and deriving compass information from celestial e-vector patterns. In the control dataset, the part of the eye of each bee conjectured to contain the receptors are painted out and the bees are exposed to a zenith patch of artificially polarized light. The resulting dance directions of 279 honeybees are available in Fisher (1995) (see page 244, Appendix B.9). The corresponding circular dotplot is shown in Fig. 3. Note that the data presented in Fisher (1995) are rounded values, with rounding done to the nearest multiple of 10° (i.e., $k = 36$). It is of interest to test the null hypothesis of circular uniformity of the dance directions for this (rounded) control dataset.

The Rao's spacing test, when applied to this rounded data without any modification yields a test statistic value of 313.5484. This corresponds to a p-value of approximately 0, strongly indicating non-uniformity. However, using the proposed modification of the rounded data, the test statistic value comes to be 2.3017, resulting in an associated p-value of 0.5391, which fails to provide significant evidence against uniformity of the dance directions. The conclusion from the modified test thus coincides with scientific theory.

5. Concluding remarks

Tests for circular uniformity based on spacings of rounded observations can have inflated Type I error probabilities if one ignores the rounding mechanism and treats the data as if they are continuous and have not been rounded. The inflation in Type I error depends on the amount of rounding and the size of the sample. In this paper, we provide a method of adjusting the data to correct for the effect of rounding, and using the adjusted data to calculate the test statistic. This adjusting of data needs knowledge of the rounding mechanism. We considered the commonly used Rao's spacing test for circular isotropy as well as the Greenwood's test. These tests maintain their Type I error probabilities perfectly when our suggested method for correcting the rounding is applied, with the advantage that the critical values currently available continue to be applicable.

It is important to note that our proposed modification to rounded or truncated data is such that it recovers the same distribution for the test statistics based on the original data under the null hypothesis of uniformity. Consequently, the test is able to maintain the same probability of Type I error (but not necessarily maintain the same power). For these spacings tests to retain the same power as before the data is rounded or truncated, we need to know the alternative under which the power is being calculated. Such an alternative can then be taken into account for our adjustment, so that the adjusted data follows the same alternative distribution as the original data before rounding. This can be quite tricky.

Recently, the issue of using Rao's spacing test in the presence of rounded data was also discussed by Landler et al. (2019). They proposed adding random perturbations of von Mises variates with zero mean and a high concentration ($\kappa = 1000$) to the rounded values, and working with Rao's spacing test calculated using the resulting modified observations. Based on limited simulation studies, they claim that their proposed modification maintains Type I error rate as well as power against the von Mises alternatives. We believe that their proposed method is not the theoretically correct way of adjusting for the effect of rounding, either under the null hypothesis of uniformity or under the von Mises alternatives. The only benefit of their method seems to be removal of ties among the observations and the resulting zero-valued spacings, which affect the Type I error.

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Data availability

Data are available from Appendix B.9 on page 244 of Fisher N.I. (1995) Statistical Analysis of Circular Data, Cambridge University Press.

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